LECTURE 7

LIMITING DISTRIBUTIONS

LIMITING DISTRIBUTIONS

- The p.d.f. of a r.v. often depends on the sample size (i.e., n)
- *If* $X_1, X_2, \ldots, X_n$ is a sequence of rvs and $Y_n = u(X_1, X_2, \ldots, X_n)$ is a function of them, sometimes it is possible to find the exact distribution of Y_n.
- However, sometimes it is only possible to obtain *approximate* results when n is large → limiting distributions.

- Limit Theorems can be used to obtain properties of estimators as the sample sizes tend to infinity
  - Convergence in Distribution – Limit of a CDF
  - Convergence in Probability – Limit of an estimator
  - Central Limit Theorem – Large Sample Distribution of the Sample Mean of a Random Sample

CONVERGENCE IN DISTRIBUTION

- Consider that $X_1, X_2, \ldots, X_n$ is a sequence of rvs and $Y_n = u(X_1, X_2, \ldots, X_n)$ be a function of rvs with cdfs $F_n(y)$ so that for each $n=1, 2, \ldots$

$$F_n(y) = P(Y_n \leq y),$$

$$\lim_{n\to\infty} F_n(y) = F(y) \quad \text{for all } y$$

where $F(y)$ is continuous. Then, the sequence Y_n is said to converge in distribution to Y.

$$Y_n \xrightarrow{d} Y$$

- Theorem: If $\lim_{n\to\infty} F_n(y) = F(y)$ for every point *y* at which <u>*F(y)* is continuous</u>, then Y_n *is said to have a* limiting distribution with cdf *F(y).* The term "Asymptotic distribution" is sometimes used instead of "limiting distribution"

- Definition of convergence in distribution requires only that limiting function agrees with cdf at its points of continuity.

Example

- Let $X_1,\ldots, X_n$ be a random sample from Unif(0,1). Find the limiting distribution of the max order statistic, if it exists.
- The CDF for Y_n is $G_n(y) = y^n \; 0 < y < 1$

$$\lim_{n\to\infty} G_n(y) = G(y) \quad for\ 0 < y < 1 \quad \lim_{n\to\infty} y^n = 0\ for\ y \le 0 \quad \lim_{n\to\infty} y^n = 0\ for\ y \ge 1 \quad \lim_{n\to\infty} y^n = 1$$

$$G(y) = \begin{cases} 0 & y < 1 \\ 1 & y \ge 1 \end{cases}$$

Example

- Let $X_1,\ldots, X_n$ be a random sample from Exp(θ). Find the limiting distribution of the min order statistic, if it exists.
- The CDF for Y_{min} is

$$G_n(y) = 1 - e^{-2y/\theta} \quad y > 0$$

$$\lim_{n\to\infty} G_n(y) = \lim_{n\to\infty} 1 - e^{-2y/\theta} = 1 \quad \textit{for } y > 0 \quad \lim_{n\to\infty} G_n(y) = 0 \ \textit{ for } y \le 0$$

$$G(y) = \begin{cases} 0 & y \le 1 \\ 1 & y > 0 \end{cases}$$

LIMITING MOMENT GENERATING FUNCTIONS

- Let rv Y_n have an mgf $M_n(t)$ that exists for all n. If

$$\lim_{n\to\infty} M_n(t) = M(t),$$

then Y_n has a limiting distribution which is defined by $M(t)$.

Example

$$X \sim Bin(n, p) \text{ with mgf } M_X(t) = (pe^t + 1 - p)^n$$

Let $\lambda = np$. Hint: $\lim_{n\to\infty} (1+a/n)^n = e^a$

$$\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty} (pe^t + 1 - p)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = e^{\lambda(e^t - 1)} = M_Y(t)$$

The mgf of Poisson(λ)

The limiting distribution of Binomial rv is the Poisson distribution.

Example

$$X \sim Poisson(\lambda) \quad \Rightarrow \quad E(X) = \lambda = V(X), \quad M_X(t) = e^{\lambda(e^t - 1)}$$

$$Y = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X - \lambda}{\sqrt{\lambda}} = aX + b \qquad a = \frac{1}{\sqrt{\lambda}}, \quad b = -\sqrt{\lambda}$$

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

$$\Rightarrow M_Y(t) = e^{-t\sqrt{\lambda}} e^{\lambda(e^{t/\sqrt{\lambda}} - 1)} = \exp\left[-t\sqrt{\lambda} + \lambda\left(e^{t/\sqrt{\lambda}} - 1 \right) \right]$$

$$\text{Use: } e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \Rightarrow e^{t/\sqrt{\lambda}} - 1 = -1 + 1 + \left(t/\sqrt{\lambda} \right) + \frac{t^2/\lambda}{2!} + \frac{t^3/\lambda^{3/2}}{3!} + \cdots$$

$$\Rightarrow M_Y(t) = \exp\left\{ -t\sqrt{\lambda} + \lambda\left(-1 + 1 + \left(t/\sqrt{\lambda} \right) + \frac{\left(t^2/\lambda \right)}{2!} + \frac{\left(t^3/\lambda^{3/2} \right)}{3!} + \cdots \right) \right\}$$

$$= \exp\left\{ -t\sqrt{\lambda} + \left(\left(t\sqrt{\lambda} \right) + \frac{\left(t^2 \right)}{2!} + \frac{\left(t^3/\lambda^{1/2} \right)}{3!} + \cdots \right) \right\} = \exp\left\{ \left(\frac{\left(t^2 \right)}{2!} + \frac{\left(t^3/\lambda^{1/2} \right)}{3!} + \cdots \right) \right\}$$

Now taking limit as $\lambda \to \infty$:

$$\lim_{\lambda \to \infty} M_Y(t) = \lim_{\lambda \to \infty} \exp\left\{ \left(\frac{\left(t^2 \right)}{2!} + \frac{\left(t^3/\lambda^{1/2} \right)}{3!} + \cdots \right) \right\} = e^{t^2/2} \equiv MGF(N(0,1))$$

CONVERGENCE IN PROBABILITY

- The sequence of random variables, $X_1,\ldots,X_n$, is said to **converge in probability** to the constant *c*, if for every $\varepsilon>0$,

$$\lim_{n\to\infty} P(|X_n - c| \le \varepsilon) = 1$$

- Weak Law of Large Numbers (WLLN): Let $X_1,\ldots,X_n$ be iid random variables with $E(X_i)=\mu$ and $V(X_i)=\sigma^2 < \infty$. Then the sample mean converges in probability to μ:

$$\lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \ge \varepsilon\right) = 0 \quad \text{or} \quad \lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \le \varepsilon\right) = 1$$

$$\text{where } \overline{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

PROOF OF WLLN

$$E\left(\overline{X}_n\right) = \mu_{\overline{X}} = \mu \quad V\left(\overline{X}_n\right) = \frac{\sigma^2}{n} \Rightarrow \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

Chebyshev's Inequality: $P(\mu_X - k\sigma_X \leq X \leq \mu_X + k\sigma_X) \geq 1 - \frac{1}{k^2} \quad (k \geq 1)$

$$\Rightarrow P(|X - \mu_X| \leq k\sigma_X) \geq 1 - \frac{1}{k^2} \quad \Rightarrow \quad P(|X - \mu_X| \geq k\sigma_X) \leq \frac{1}{k^2}$$

$$\Rightarrow P\left(|\overline{X}_n - \mu_{\overline{X}}| \geq k\sigma_{\overline{X}} = \frac{k\sigma}{\sqrt{n}}\right) \leq \frac{1}{k^2}$$

Let: $\varepsilon = \frac{k\sigma}{\sqrt{n}} \quad \Rightarrow \quad k = \frac{\sqrt{n}\varepsilon}{\sigma} \Rightarrow k^2 = \frac{n\varepsilon^2}{\sigma^2} \Rightarrow \frac{1}{k^2} = \frac{\sigma^2}{n\varepsilon^2}$

$$\Rightarrow P\left(|\overline{X}_n - \mu_{\overline{X}}| \geq k\sigma_{\overline{X}} = \frac{k\sigma}{\sqrt{n}} = \varepsilon\right) \leq \frac{1}{k^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} P\left(|\overline{X}_n - \mu_{\overline{X}}| \geq \varepsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0 \quad \forall \varepsilon > 0$$

$$\Rightarrow \overline{X}_n \xrightarrow{\text{Prob}} \mu$$

- Binomial Sample Proportions

$$X \sim \text{Binomial}(n, p) \quad X_i = \begin{cases} 1 & \text{if Trial } i \text{ is a Success} \\ 0 & \text{if Trial } i \text{ is a Failure} \end{cases} \quad E(X_i) = p \quad V(X_i) = p(1-p)$$

$$X = \sum_{i=1}^{n} X_i \Rightarrow E(X) = np, \quad V(X) = np(1-p)$$

$$\text{Let} \quad \hat{p} = \frac{X}{n} = \frac{\sum_{i=1}^{n} X_i}{n} \Rightarrow E\left(\hat{p}\right) = p, \quad V\left(\hat{p}\right) = \frac{p(1-p)}{n}$$

$$\Rightarrow \hat{p} \xrightarrow{\text{Prob}} p$$

STRONG LAW OF LARGE NUMBERS

- Let X_1, $X_2,\ldots,X_n$ be iid rvs with $E(X_i)=\mu$ and $V(X_i)=\sigma^2<\infty$. Define $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$. Then, for every $\varepsilon>0$,

$$P\left(\lim_{n\to\infty}\left|\bar{X}_n-\mu\right|<\varepsilon\right)=1$$

that is, $\bar{X}_n$ converges almost surely to μ.

RELATION BETWEEN CONVERGENCES

Almost sure convergence is the strongest.

$$Y_n \xrightarrow{a.s} Y \;\Rightarrow Y_n \xrightarrow{p} Y \;\Rightarrow Y_n \xrightarrow{d} Y$$

(reverse is generally not true)

THE CENTRAL LIMIT THEOREM

- Let $X_1, X_2, \ldots, X_n$ be a sequence of iid rvs with $E(X_i)=\mu$ and $V(X_i)=\sigma^2<\infty$. Define $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$
- Then,

$$Z = \frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \xrightarrow{d} N(0,1)$$

or

$$Z = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1).$$

Example

- Let $X_1, X_2, \ldots, X_n$ be iid rvs from Unif(0,1) and

$$Y_n = \sum_{i=1}^{n} X_i$$

- $E(X_i)$=1/2 and $Var(X_i)$=1/12
- $Y_n \sim N(n/2, n/12)$

Examples

- A very common application of the CLT is the Normal approximation to the Binomial distribution.

- Suppose X_1, X_2,…are i.i.d random variables and each has the Bernoulli(p) distribution. So $E(X_i) = p$ and $V(X_i) = p(1- p)$.

- The CLT says that $P\left(X_1 + \cdots + X_n \le np + x\sqrt{np(1-p)}\right) \to \Phi(x)$ as $n \to \infty$.

- Let $Y_n = X_1 + \ldots + X_n$ then Y_n has a Binomial(n, p) distribution.

So for large n,

$$P(Y_n \le y) = P\left(\frac{Y_n - np}{\sqrt{np(1-p)}} \le \frac{y - np}{\sqrt{np(1-p)}}\right) \approx \Phi\left(\frac{y - np}{\sqrt{np(1-p)}}\right)$$

SLUTKY'S THEOREM

- If $X_n \to X$ in distribution and $Y_n \to a$, a constant, in probability, then

  a) $Y_n X_n \to aX$ in distribution.

  b) $X_n + Y_n \to X + a$ in distribution.

SOME THEOREMS ON LIMITING DISTRIBUTIONS

- If $X_n \to c>0$ in probability, then for any function g(x) continuous at c, g(X_n) $\to g(c)$ in prob. e.g.

$$\sqrt{X_n} \xrightarrow{p} \sqrt{c}.$$

- If $X_n \to c$ in probability and $Y_n \to d$ in probability, then

  - $aX_n+bY_n \to ac+bd$ in probability.
  - $X_nY_n \to cd$ in probability
  - $1/X_n \to 1/c$ in probability for all $c \neq 0$.

Example

- Consider a random sample size of n from Bernoulli distribution $X_i \sim Bin(1, p)$

$$\frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \xrightarrow{d} Z \sim N(0,1)$$

$$Y \sim Bin(n, p) \quad \hat{p} = Y/n \xrightarrow{P} p \quad \hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$$

$$\sqrt{\hat{p}(1-\hat{p})/\, p(1-p)}$$

$$\frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})/n}} \xrightarrow{d} Z \sim N(0,1)$$