LECTURE 5

JOINT, MARGINAL, CONDITIONAL DISTRIBUTION

MULTIVARIATE RANDOM VARIABLES

- In many applications there will be more than one random variable $X_1, X_2, \ldots, X_k$
- Often used to study the relationship among characteristics and the prediction of one based on the other(s)
- Three types of distributions:
  - Joint: Distribution of outcomes across all combinations of variables levels
  - Marginal: Distribution of outcomes for a single variable
  - Conditional: Distribution of outcomes for a single variable, given the level(s) of the other variable(s)

JOINT DISCRETE DISTRIBUTION

Let $X_1, X_2, \ldots, X_k$ denote k discrete random variables, then

$$p(x_1, x_2, \ldots, x_k)$$

is joint probability function of $X_1, X_2, \ldots, X_k$ if

1. $0 \le p\left(x_1, \ldots, x_n\right) \le 1$

2. $\sum_{x_1} \cdots \sum_{x_n} p\left(x_1, \ldots, x_n\right) = 1$

3. $P\left[\left(X_1, \ldots, X_n\right) \in A\right] = \sum \cdots \sum_{\left(x_1, \ldots, x_n\right) \in A} p\left(x_1, \ldots, x_n\right)$

Example

- For e.g.: Tossing two fair dice → 36 possible sample points
- Let X: sum of the two dice;

  Y: difference of the two dice

  – For (3,3), X=6 and Y=0.

  – For both (4,1) and (1,4), X=5, Y=3.

- Joint pmf (pdf) of (x,y)

		x										
		2	3	4	5	6	7	8	9	10	11	12
	0	1/36		1/36		1/36		1/36		1/36		1/36
	1		1/18		1/18		1/18		1/18		1/18	
y	2			1/18		1/18		1/18		1/18		
	3				1/18		1/18		1/18			
	4					1/18		1/18				
	5						1/18					

Empty cells are equal to 0.

e.g. $P(X=7,Y\leq 4)=f(7,0)+f(7,1)+f(7,2)+f(7,3)+f(7,4)=0+1/18+0+1/18+0=1/9$

Example

- A bridge hand (13 cards) is selected from a deck of 52 cards.
- X = the number of spades in the hand.
- Y = the number of hearts in the hand.
- In this example we will define:
- $p(x,y) = P[X = x, Y = y]$

(Extended Hypergeometric Distribution)

Note:

The possible values of X are 0, 1, 2, …, 13

The possible values of Y are also 0, 1, 2, …, 13 and $X + Y \leq 13$.

$$p(x, y) = P(X = x, Y = y)$$

$$= \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$$

- $\binom{13}{x}$: The number of ways of choosing the x **spades** for the hand
- $\binom{13}{y}$: The number of ways of choosing the y **hearts** for the hand
- $\binom{26}{13-x-y}$: The number of ways of completing the hand
- $\binom{52}{13}$: The total number of ways of choosing the 13 cards for the hand

$$p(x,y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0.0000	0.0002	0.0009	0.0024	0.0035	0.0032	0.0018	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0002	0.0021	0.0085	0.0183	0.0229	0.0173	0.0081	0.0023	0.0004	0.0000	0.0000	0.0000	0.0000	-
2	0.0009	0.0085	0.0299	0.0549	0.0578	0.0364	0.0139	0.0032	0.0004	0.0000	0.0000	0.0000	-	-
3	0.0024	0.0183	0.0549	0.0847	0.0741	0.0381	0.0116	0.0020	0.0002	0.0000	0.0000	-	-	-
4	0.0035	0.0229	0.0578	0.0741	0.0530	0.0217	0.0050	0.0006	0.0000	0.0000	-	-	-	-
5	0.0032	0.0173	0.0364	0.0381	0.0217	0.0068	0.0011	0.0001	0.0000	-	-	-	-	-
6	0.0018	0.0081	0.0139	0.0116	0.0050	0.0011	0.0001	0.0000	-	-	-	-	-	-
7	0.0006	0.0023	0.0032	0.0020	0.0006	0.0001	0.0000	-	-	-	-	-	-	-
8	0.0001	0.0004	0.0004	0.0002	0.0000	0.0000	-	-	-	-	-	-	-	-
9	0.0000	0.0000	0.0000	0.0000	0.0000	-	-	-	-	-	-	-	-	-
10	0.0000	0.0000	0.0000	0.0000	-	-	-	-	-	-	-	-	-	-
11	0.0000	0.0000	0.0000	-	-	-	-	-	-	-	-	-	-	-
12	0.0000	0.0000	-	-	-	-	-	-	-	-	-	-	-	-
13	0.0000	-	-	-	-	-	-	-	-	-	-	-	-	-

Example

Suppose that we observe an experiment that has k possible outcomes $\{O_1, O_2, \ldots, O_k\}$ independently n times.

Let $p_1, p_2, \ldots, p_k$ denote probabilities of $O_1, O_2, \ldots, O_k$ respectively.

Let X_i denote the number of times that outcome O_i occurs in the n repetitions of the experiment.

Then the joint probability function of the random variables $X_1, X_2, \ldots, X_k$ is (The Multinomial distribution)

$$p\left(x_1, \ldots, x_n\right) = \frac{n!}{x_1!\, x_2! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

Note:

$$p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

is the probability of a sequence of length n containing

x_1 outcomes O_1

x_2 outcomes O_2

…

x_k outcomes O_k

$$\frac{n!}{x_1!x_2!\ldots x_k!} = \begin{pmatrix} & n & & \\ x_1 & x_2 & \cdots & x_k \end{pmatrix}$$

is the number of ways of choosing the positions for the x_1 outcomes O_1, x_2 outcomes O_2, …, x_k outcomes O_k

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}\ldots\binom{x_k}{x_k}$$

$$= \left(\frac{n!}{x_1! \; n-x_1 \; !}\right)\left(\frac{n-x_1 \; !}{x_2! \; n-x_1-x_2 \; !}\right)\left(\frac{n-x_1-x_2 \; !}{x_3! \; n-x_1-x_2-x_3 \; !}\right)\ldots$$

$$= \frac{n!}{x_1!x_2!\ldots x_k!}$$

$$p\left(x_1, \ldots, x_n\right) = \frac{n!}{x_1!\,x_2!\ldots x_k!}\, p_1^{x_1}\, p_2^{x_2} \ldots p_k^{x_k}$$

$$= \binom{n}{x_1 \quad x_2 \quad \ldots \quad x_k} p_1^{x_1}\, p_2^{x_2} \ldots p_k^{x_k}$$

is called the **Multinomial** distribution

Suppose that a treatment for back pain has three possible outcomes:

O_1 - Complete cure (no pain) – (30% chance)

O_2 - Reduced pain – (50% chance)

O_3 - No change – (20% chance)

Hence $p_1 = 0.30$, $p_2 = 0.50$, $p_3 = 0.20$.

Suppose the treatment is applied to $n = 4$ patients suffering back pain and let X = the number that result in a complete cure, Y = the number that result in just reduced pain, and Z = the number that result in no change.

Find the distribution of *X, Y* and *Z.*

$$p \quad x, y, z \quad = \frac{4!}{x!y!z!} \; 0.30^{\;x} \; 0.50^{\;y} \; 0.20^{\;z} \qquad x + y + z = 4$$

Table: $p(x,y,z)$

x	y	z = 0	1	2	3	4
0	0	0	0	0	0	0.0016
0	1	0	0	0	0.0160	0
0	2	0	0	0.0600	0	0
0	3	0	0.1000	0	0	0
0	4	0.0625	0	0	0	0
1	0	0	0	0	0.0096	0
1	1	0	0	0.0720	0	0
1	2	0	0.1800	0	0	0
1	3	0.1500	0	0	0	0
1	4	0	0	0	0	0
2	0	0	0	0.0216	0	0
2	1	0	0.1080	0	0	0
2	2	0.1350	0	0	0	0
2	3	0	0	0	0	0
2	4	0	0	0	0	0
3	0	0	0.0216	0	0	0
3	1	0.0540	0	0	0	0
3	2	0	0	0	0	0
3	3	0	0	0	0	0
3	4	0	0	0	0	0
4	0	0.0081	0	0	0	0
4	1	0	0	0	0	0
4	2	0	0	0	0	0
4	3	0	0	0	0	0
4	4	0	0	0	0	0

MARGINAL DISCRETE DISTRIBUTIONS

Let $X_1, X_2, \ldots, X_q, X_{q+1} \ldots, X_k$ denote k discrete random variables with joint probability function

$$p(x_1, x_2, \ldots, x_q, x_{q+1} \ldots, x_k)$$

then the marginal joint probability function of $X_1, X_2, \ldots, X_q$ is

$$p_{12\ldots q}\left(x_1, \ldots, x_q\right) = \sum_{x_{q+1}} \cdots \sum_{x_n} p\left(x_1, \ldots, x_n\right)$$

- If the pair (X_1,X_2) of discrete random variables has the joint pmf $p(x_1,x_2)$, then the marginal pmfs of X_1 and X_2 are

$$p_1(x_1) = \sum_{x_2} p(x_1, x_2) \quad \text{and} \quad p_2(x_2) = \sum_{x_1} p(x_1, x_2)$$

Example

A die is rolled n = 5 times

X = the number of times a "**six**" appears.

Y = the number of times a "**five**" appears.

	0	1	2	3	4	5	$p_X(x)$
0	0.1317	0.1646	0.0823	0.0206	0.0026	0.0001	0.4019
1	0.1646	0.1646	0.0617	0.0103	0.0006	0	0.4019
2	0.0823	0.0617	0.0154	0.0013	0	0	0.1608
3	0.0206	0.0103	0.0013	0	0	0	0.0322
4	0.0026	0.0006	0	0	0	0	0.0032
5	0.0001	0	0	0	0	0	0.0001
$p_Y(y)$	0.4019	0.4019	0.1608	0.0322	0.0032	0.0001	

Example

Assume that the random variables X and Y have the joint probability mass function given as

$$f(x,y) = \frac{\lambda^{x+y} e^{-2\lambda}}{x!y!} \qquad x = 0, 1, 2, .. \quad y = 0,1,2,......$$

Find the marginal distribution of X

$$f(x) = \sum \frac{\lambda^{x+y} e^{-2\lambda}}{x!\, y!} = \sum_{Y=0}^{\infty} \frac{\lambda^{Y}}{Y!} \frac{\lambda^{x}}{X!} e^{-2\lambda}$$

$$= \frac{\lambda^{x} e^{-\lambda}}{x!}$$

[Using $e^{\lambda} = \sum_{t=0}^{\infty} \frac{\lambda^t}{t!}$]

Example

Let the joint distribution of X and Y be given as

$$f(x\,,y) = \frac{x+y}{30} \qquad x = 0,l,2,3 \quad y = 0,1,2,$$

Find the marginal distribution function of X and Y. Marginal of X

$$f(x) = \sum_{y=0}^{2} f(x,y) = \sum_{y=0}^{2} \frac{x+y}{30} = \frac{1}{30}[(x+0)+(x+1)+(x+2)] = \frac{1}{30}[3x+3] = \frac{x+1}{10}$$

Similarly, f(y) = (3+2x)/15

<u>Or , since the joint is</u>

y \ x	0	1	2	3
0	0	1/30	1/15	1/10
1	1/30	1/15	1/10	2/15
2	1/15	1/10	2/15	1/6
	1/10	1/5	3/10	2/5

x	0	1	2	3
f(x)	1/10	1/5	3/10	2/5

Y	0	1	2
f(y)	1/5	1/3	7/15

JOINT DISCRETE CUMULATIVE DISTRIBUTION FUNCTION

$$F(x_1, x_2, \ldots, x_k) = P(X_1 \le x_1, \ldots, X_k \le x_k)$$

- $F(x_1,x_2)$ is a cdf iff

$$\lim_{x_1 \to -\infty} F(x_1, x_2) = F(-\infty, x_2) = 0, \forall x_2.$$

$$\lim_{x_2 \to -\infty} F(x_1, x_2) = F(x_1, -\infty) = 0, \forall x_1.$$

$$\lim_{\substack{x_1 \to \infty \\ x_2 \to \infty}} F(x_1, x_2) = F(\infty, \infty) = 1$$

$$P(a < X_1 \le b, c < X_2 \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c) \ge 0, \forall \text{ a} < \text{b and c} < \text{d}.$$

$$\lim_{h \to 0^+} F(x_1 + h, x_2) = \lim_{h \to 0^+} F(x_1, x_2 + h) = F(x_1, x_2), \forall \text{ x}_1 \; and \; \text{x}_2.$$

CONDITIONAL DISCRETE DISTRIBUTION

Let $X_1, X_2, \ldots, X_q, X_{q+1} \ldots, X_k$ denote *k* discrete random variables with joint probability function

$$p(x_1, x_2, \ldots, x_q, x_{q+1} \ldots, x_k)$$

then the **conditional** joint probability function of $X_1, X_2, \ldots, X_q$ given $X_{q+1} = x_{q+1}, \ldots, X_k = x_k$ is

$$p_{1\ldots q|q+1\ldots k}\left(x_1,\ldots,x_q \middle| x_{q+1},\ldots,x_k\right) = \frac{p\left(x_1,\ldots,x_k\right)}{p_{q+1\ldots k}\left(x_{q+1},\ldots,x_k\right)}$$

Let X and Y denote two discrete random variables with joint probability function

$$p(x,y) = P[X = x, Y = y]$$

Then

$p_{X|Y}(x|y) = P[X = x|Y = y]$ is called the **conditional probability function** of X given $Y = y$

and

$p_{Y|X}(y|x) = P[Y = y|X = x]$ is called the **conditional probability function** of Y given $X = x$

Note

$$p_{X|Y}(x|y) = P\left[X = x | Y = y\right]$$

$$= \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}$$

and

$$p_{Y|X}(y|x) = P\left[Y = y | X = x\right]$$

$$= \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)}$$

Example

A die is rolled $n = 5$ times

X = the number of times a "**six**" appears.

Y = the number of times a "**five**" appears.

		y					
	0	**1**	**2**	**3**	**4**	**5**	$p_X(x)$
x = 0	0.1317	0.1646	0.0823	0.0206	0.0026	0.0001	0.4019
1	0.1646	0.1646	0.0617	0.0103	0.0006	0	0.4019
2	0.0823	0.0617	0.0154	0.0013	0	0	0.1608
3	0.0206	0.0103	0.0013	0	0	0	0.0322
4	0.0026	0.0006	0	0	0	0	0.0032
5	0.0001	0	0	0	0	0	0.0001
$p_Y(y)$	0.4019	0.4019	0.1608	0.0322	0.0032	0.0001	

The conditional distribution of *X* given *Y* = *y*.

$$p_{X\,|\,Y}(x\,|\,y) = P[X = x\,|\,Y = y]$$

x \ y	0	1	2	3	4	5
0	0.3277	0.4096	0.5120	0.6400	0.8000	1.0000
1	0.4096	0.4096	0.3840	0.3200	0.2000	0.0000
2	0.2048	0.1536	0.0960	0.0400	0.0000	0.0000
3	0.0512	0.0256	0.0080	0.0000	0.0000	0.0000
4	0.0064	0.0016	0.0000	0.0000	0.0000	0.0000
5	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000

The conditional distribution of Y given $X = x$.

$$p_{Y|X}(y|x) = P[Y = y|X = x]$$

x \ y	0	1	2	3	4	5
0	0.3277	0.4096	0.2048	0.0512	0.0064	0.0003
1	0.4096	0.4096	0.1536	0.0256	0.0016	0.0000
2	0.5120	0.3840	0.0960	0.0080	0.0000	0.0000
3	0.6400	0.3200	0.0400	0.0000	0.0000	0.0000
4	0.8000	0.2000	0.0000	0.0000	0.0000	0.0000
5	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example

The joint probability mass function of X and Y is given by

$f(1,1) = \frac{1}{8}$ $f(1,2) = \frac{1}{4}$ $f(2,1)= \frac{1}{8}$ $f(2,2)= \frac{1}{2}$

1.Compute the conditional mass function of X given Y = i, i =1,2

2.Compute $P(XY \le 3) = f(1, 1) + f(1, 2) + f(2, 1) = 1/2$

3. $P(X/Y> l) = f(2, 1)= 1/8$

Y \ x	1	2	Sum
1	1/8	1/8	2/8
2	1/4	1/2	6/8
sum	3/8	5/8	1

Marginal of y

y	1	2
f(y)	2/8	6/8

The conditional mass f^{n} of X/Y =1

x	1	2
f(x\y=1)	1/2	1/2

The conditional of X/Y=2

y	1	2	Sum
f(x\y=2)	1/3	2/3	1

JOINT CONTINUOUS DISTRIBUTION

Let $X_1, X_2, \ldots, X_k$ denote k continuous random variables, then

$$f(x_1, x_2, \ldots, x_k)$$

is joint density function of $X_1, X_2, \ldots, X_k$ if

1. $f\left(x_1, \ldots, x_n\right) \geq 0$
2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_1, \ldots, x_n\right) dx_1, \ldots, dx_n = 1$
3. $P\left[\left(X_1, \ldots, X_n\right) \in A\right] = \int \cdots \int_A f\left(x_1, \ldots, x_n\right) dx_1, \ldots, dx_n$

Example

- Assume that joint pdf of random variable X and Y is $f(x, y) = 4\,\mathrm{xy} \quad 0 < \mathrm{x} < 1, \quad 0 < \mathrm{y} < 1$ the joint cdf is

$$F(x, y) = \int_0^x \int_0^y 4 t_1 t_2 \, dt_1 dt_2 = x^2 y^2 \quad 0 < \mathrm{x} < 1, \quad 0 < \mathrm{y} < 1$$

MARGINAL CONTINUOUS DISTRIBUTION

Let $X_1, X_2, \ldots, X_q, X_{q+1} \ldots, X_k$ denote k continuous random variables with joint probability density function

$$f(x_1, x_2, \ldots, x_q, x_{q+1} \ldots, x_k)$$

then the marginal joint probability function of $X_1, X_2, \ldots, X_q$ is

$$f_{12\ldots q}\left(x_1,\ldots,x_q\right) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f\left(x_1,\ldots,x_n\right) dx_{q+1}\ldots dx_n$$

- If the pair (X_1,X_2) of discrete random variables has the joint pdf $f(x_1,x_2)$, then the marginal pdfs of X_1 and X_2 are

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 \; and \; f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1.$$

Example

Joint density f(x,y) for X and Y:

$$f(x,y)=\begin{cases}\frac{2}{5}(2x+3y) & 0\le x\le 1,\, 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Marginal density function for X:

$$g(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_{0}^{1}\frac{2}{5}(2x+3y)\,dy$$

$$=\left[\frac{2}{5}2xy+\frac{1}{5}3y^{2}\right]_{0}^{1}=\frac{4}{5}x+\frac{3}{5}$$

INDEPENDENCE OF RANDOM VARIABLES

- If $X_1, X_2, \ldots, X_k$ are independent from each other, then the joint pdf can be given as

$$f(x_1, x_2, \ldots, x_k) = f(x_1)\, f(x_2) \ldots f(x_k)$$

And the joint cdf can be written as

$$F(x_1, x_2, \ldots, x_k) = F(x_1)\, F(x_2) \ldots F(x_k)$$

CONTINUOUS CONDITIONAL DISTRIBUTIONS

Let $X_1, X_2, \ldots, X_q, X_{q+1} \ldots, X_k$ denote k continuous random variables with joint probability density function

$$f(x_1, x_2, \ldots, x_q, x_{q+1} \ldots, x_k)$$

then the **conditional** joint probability function

of $X_1, X_2, \ldots, X_q$ given $X_{q+1} = x_{q+1}, \ldots, X_k = x_k$ is

$$f_{1\ldots q|q+1\ldots k}\left(x_1, \ldots, x_q \middle| x_{q+1}, \ldots, x_k\right) = \frac{f\left(x_1, \ldots, x_k\right)}{f_{q+1\ldots k}\left(x_{q+1}, \ldots, x_k\right)}$$

- If X_1 and X_2 are continuous random variables with joint pdf $f(x_1,x_2)$, then the conditional pdf of X_2 given $X_1=x_1$ is defined by

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)}, \forall x_1 \text{ such that } f(x_1) > 0, 0 \text{ elsewhere.}$$

- For independent rvs,

$$f(x_2|x_1) = f(x_2).$$
$$f(x_1|x_2) = f(x_1).$$

- If X and Y are random variables with joint pdf $f(x,y)$ and marginal pdf's $f(x), f(y)$

$$f(x,y) = f(x)f(y|x) = f(y)f(x|y)$$

and if X and Y are independent then

$$f(x|y) = f(x)$$
$$f(y|x) = f(y)$$

Example

Suppose that a rectangle is constructed by first choosing its length, X and then choosing its width Y.

Its length X is selected form an exponential distribution with mean $\mu = {}^{1}/_{\lambda} = 5$. Once the length has been chosen its width, Y, is selected from a uniform distribution form 0 to half its length.

Find the joint distribution function.

$$f_X(x) = \tfrac{1}{5} e^{-\frac{1}{5}x} \quad \text{for } x \geq 0$$

$$f_{Y|X}(y|x) = \frac{1}{x/2} \quad \text{if } 0 \leq y \leq x/2$$

$$f(x,y) = f_X(x)\, f_{Y|X}(y|x)$$

$$= \tfrac{1}{5} e^{-\frac{1}{5}x} \frac{1}{x/2} = \tfrac{2}{5x} e^{-\frac{1}{5}x} \quad \text{if } 0 \leq y \leq x/2,\, x \geq 0$$

Example

Let *X*, *Y*, Z denote 3 jointly distributed random variable with joint density function then

$$f\left(x, y, z\right) = \begin{cases} K\left(x^2 + yz\right) & 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of *K*.

Determine the marginal distributions of *X*, *Y* and *Z*.

Determine the joint marginal distributions of

X, *Y*

X, *Z*

Y, *Z*

Determining the value of *K*.

$$1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x,y,z\right)\,dxdydz=\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}K\left(x^{2}+yz\right)\,dxdydz$$

$$=K\int_{0}^{1}\int_{0}^{1}\left[\frac{x^{3}}{3}+xyz\right]_{x=0}^{x=1}dydz=K\int_{0}^{1}\int_{0}^{1}\left(\frac{1}{3}+yz\right)dydz$$

$$=K\int_{0}^{1}\left[\frac{1}{3}y+z\frac{y^{2}}{2}\right]_{y=0}^{y=1}dz=K\int_{0}^{1}\left(\frac{1}{3}+z\frac{1}{2}\right)dz$$

$$=K\left[\frac{z}{3}+\frac{z^{2}}{4}\right]_{0}^{1}=K\left(\frac{1}{3}+\frac{1}{4}\right)=K\frac{7}{12}=1\quad\text{if }\;K=\frac{12}{7}$$

The marginal distribution of *X*.

$$f_1(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)\, dydz = \frac{12}{7}\int_0^1\int_0^1 (x^2 + yz)\, dydz$$

$$= \frac{12}{7}\int_0^1 \left[x^2 y + \frac{y^2}{2} z \right]_{y=0}^{y=1} dz = \frac{12}{7}\int_0^1 \left(x^2 + \frac{1}{2} z \right) dz$$

$$= \frac{12}{7}\left[x^2 z + \frac{z^2}{4} \right]_0^1 = \frac{12}{7}\left(x^2 + \frac{1}{4} \right) \quad \text{for } 0 \le x \le 1$$

The marginal distribution of *X*,*Y*.

$$f_{12}(x, y) = \int_{-\infty}^{\infty} f(x, y, z)\, dz = \frac{12}{7}\int_{0}^{1} (x^2 + yz)\, dz$$

$$= \frac{12}{7}\left[x^2 z + y\frac{z^2}{2}\right]_{z=0}^{z=1}$$

$$= \frac{12}{7}\left(x^2 + \frac{1}{2}y\right) \quad \text{for } 0 \le x \le 1, 0 \le y \le 1$$

The marginal distribution of X,Y.

$$f_{12}\left(x, y\right) = \frac{12}{7}\left(x^2 + \frac{1}{2}y\right) \quad \text{for } \; 0 \le x \le 1, 0 \le y \le 1$$

Thus the conditional distribution of Z given $X = x, Y = y$ is

$$\frac{f\left(x, y, z\right)}{f_{12}\left(x, y\right)} = \frac{\frac{12}{7}\left(x^2 + yz\right)}{\frac{12}{7}\left(x^2 + \frac{1}{2}y\right)}$$

$$= \frac{x^2 + yz}{x^2 + \frac{1}{2}y} \quad \text{for } \; 0 \le z \le 1$$

The marginal distribution of X.

$$f_1(x) = \frac{12}{7}\left(x^2 + \frac{1}{4}\right) \text{ for } 0 \le x \le 1$$

Thus the conditional distribution of Y, Z given $X = x$ is

$$\frac{f(x, y, z)}{f_1(x)} = \frac{\frac{12}{7}\left(x^2 + yz\right)}{\frac{12}{7}\left(x^2 + \frac{1}{4}\right)}$$

$$= \frac{x^2 + yz}{x^2 + \frac{1}{4}} \text{ for } 0 \le y \le 1, 0 \le z \le 1$$

Example (Bivariate Normal Distribution)

- A pair of continuous rvs X and Y is said to have a bivariate normal distribution if it has a joint pdf of the form

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_1}\right)\left(\frac{y-\mu_y}{\sigma_2}\right) + \left(\frac{y-\mu_y}{\sigma_2}\right)^2\right]\right\}$$

$$-\infty < x < \infty, -\infty < y < \infty, \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1.$$

If $(X,Y) \sim BVN(\mu_x, \mu_y, \sigma_1^2, \sigma_2^2, \rho)$, then

$$X \sim N(\mu_x, \sigma_1^2) \text{ and } Y \sim N(\mu_y, \sigma_2^2)$$

and ρ is the correlation coefficient btw X and Y.

1. Conditional on X=x,

$$Y|x \sim N\left(\mu_y + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_X), \sigma_2^2(1-\rho^2)\right)$$

2. Conditional on Y=y,

$$X|y \sim N\left(\mu_x + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_Y), \sigma_1^2(1-\rho^2)\right)$$

CONDITIONAL EXPECTATION

- For *X, Y* discrete random variables, the conditional expectation of *Y* given *X = x* is

$$E(Y \mid X = x) = \sum_y y \cdot p_{Y|X}(y \mid x)$$

and the conditional variance of *Y* given *X = x* is

$$V(Y \mid X = x) = \sum_y \left(y - E(Y \mid X = x)\right)^2 \cdot p_{Y|X}(y \mid x)$$

$$= E(Y^2 \mid X = x) - \left(E(Y \mid X = x)\right)^2$$

where these are defined only if the sums converges absolutely.

- In general, $E(h(Y) \mid X = x) = \sum_y h(y) \cdot p_{Y|X}(y \mid x)$

- For *X, Y* continuous random variables, the conditional expectation of *Y* given *X = x* is

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y \mid x)\,dy$$

and the conditional variance of *Y* given *X = x* is

$$V(Y \mid X = x) = \int_{-\infty}^{\infty} \left[y - E(Y \mid X = x)\right]^2 \cdot f_{Y|X}(y \mid x)\,dy$$

$$= E(Y^2 \mid X = x) - \left(E(Y \mid X = x)\right)^2$$

- In general, $E(h(Y) \mid X = x) = \int_y h(y) \cdot f_{Y|X}(y \mid x)\,dy$

- If X and Y are jointly distributed random variables then,

$$E[E(Y|X)]=E(Y)$$

- If X and Y are independent random variables then,

$$E(Y|X)=E(Y)$$

$$E(X|Y)=E(X)$$

- If X and Y are jointly distributed random variables then,

$$Var(Y)=E_X\ [Var(Y|X)]+Var_X[E(X \mid Y)]$$

MOMENT GENERATING FUNCTION OF A JOINT DISTRIBUTION

- The E(e^tX) and M'(0) approaches both work

$$M_{X,Y}(t_1,t_2) = E[e^{t_1X+t_2Y}]$$

$$E[X^nY^m] = \frac{d^{n+m}}{d^nt_1d^mt_2}M_{X,Y}(t_1,t_2)\,|_{t_1=t_2=0}$$

- Can get E(XY), Cov(X,Y)