IAM 530
ELEMENTS OF PROBABILITY AND STATISTICS

LECTURE 4-SOME DISCERETE AND CONTINUOUS DISTRIBUTION FUNCTIONS

SOME DISCRETE PROBABILITY DISTRIBUTIONS

Degenerate, Uniform, Bernoulli, Binomial, Poisson, Negative Binomial, Geometric, Hypergeometric

DEGENERATE DISTRIBUTION

- An rv X is degenerate at point k if

$$P\left(X=x\right)=\begin{cases}1, & X=k\\ 0, & o.w.\end{cases}$$

The cdf:

$$F\left(x\right)=P\left(X\le x\right)=\begin{cases}0, & X<k\\ 1, & X\ge k\end{cases}$$

UNIFORM DISTRIBUTION

- A finite number of equally spaced values are equally likely to be observed.

$$P(X = x) = \frac{1}{N};\ x = 1,2,\ldots,\ N; \quad N = 1,2,\ldots$$

- Example: throw a fair die. P(X=1)=…=P(X=6)=1/6

$$E(X) = \frac{N+1}{2}; \quad Var(X) = \frac{(N+1)(N-1)}{12}$$

BERNOULLI DISTRIBUTION

- An experiment consists of one trial. It can result in one of 2 outcomes: Success or Failure (or a characteristic being Present or Absent).
- Probability of Success is *p* (0<*p*<1)

$$X = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases} ;0 \le p \le 1$$

$$\mathrm{P(X = x) = p^x (1-p)^{1-x} \ for\ x = 0,1;\ and\ 0 < p < 1}$$

$$E(X) = \sum_{y=0}^{1} xp(y) = 0(1-p) + 1p = p$$

$$E\left(X^2\right) = 0^2(1-p) + 1^2 p = p$$

$$V(X) = E\left(X^2\right) - \left[E(X)\right]^2 = p - p^2 = p(1-p)$$

$$\sigma = \sqrt{p(1-p)}$$

Binomial Experiment

- Experiment consists of a series of n identical trials
- Each trial can end in one of 2 outcomes: Success or Failure
- Trials are independent (outcome of one has no bearing on outcomes of others)
- Probability of Success, p, is constant for all trials
- Random Variable X, is the number of Successes in the n trials is said to follow Binomial Distribution with parameters n and p
- X can take on the values $x=0,1,\ldots,n$
- Notation: $X \sim Bin(n,p)$

Consider outcomes of an experiment with 3 Trials :

$$SSS \Rightarrow y = 3 \quad P(SSS) = P(Y = 3) = p(3) = p^3$$

$$SSF, SFS, FSS \Rightarrow y = 2 \quad P(SSF \cup SFS \cup FSS) = P(Y = 2) = p(2) = 3p^2(1-p)$$

$$SFF, FSF, FFS \Rightarrow y = 1 \quad P(SFF \cup FSF \cup FFS) = P(Y = 1) = p(1) = 3p(1-p)^2$$

$$FFF \Rightarrow y = 0 \quad P(FFF) = P(Y = 0) = p(0) = (1-p)^3$$

In General :

1) # of ways of arranging $x\ S^s$ (and $(n-x)\ F^s$) in a sequence of n positions $\equiv \binom{n}{x} = \frac{n!}{x!(n-x)!}$

2) Probability of each arrangement of $x\ S^s$ (and $(n-x)\ F^s$) $\equiv p^x(1-p)^{n-x}$

3) $P(X = x) = p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n$

- **Example:**
- There are black and white balls in a box. Select and record the color of the ball. Put it back and re-pick (sampling *with* replacement).
- n: number of independent and identical trials
- p: probability of success (e.g. probability of picking a black ball)
- X: number of successes in n trials

BINOMIAL THEOREM

- For any real numbers x and y and integer $n>0$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

- If X~Bin(n,p), then

$$E(X) = np$$

$$Var(X) = np(1 - p)$$

POISSON DISTRIBUTION

- The number of occurrences in a given time interval can be modeled by the Poisson distribution.
- e.g. waiting for bus, waiting for customers to arrive in a bank.

- Another application is in spatial distributions.
- e.g. modeling the distribution of bomb hits in an area or the distribution of fish in a lake.

POISSON DISTRIBUTION

- If X~ Poisson(λ), then $$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0,1,2,\ldots$$

- E(X)= Var(X)=λ

RELATIONSHIP BETWEEN BINOMIAL AND POISSON

$$X \sim Bin(n, p) \quad \text{with mgf } M_X(t) = \left(pe^t + 1 - p\right)^n$$

Let $\lambda = np$.

$$\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty}\left(pe^t + 1 - p\right)^n$$

$$= \lim_{n\to\infty}\left(1 + \frac{\lambda\left(e^t - 1\right)}{n}\right)^n = e^{\lambda\left(e^t - 1\right)} = M_Y(t)$$

The mgf of Poisson(λ)

The limiting distribution of Binomial rv is the Poisson distribution.

NEGATIVE BINOMIAL DISTRIBUTION (PASCAL OR WAITING TIME DISTRIBUTION)

- X: number of Bernoulli trials required to get a fixed number of *failures* before the r th success; or, alternatively,
- Y: number of Bernoulli trials required to get a fixed number of *successes*, such as r successes.

NEGATIVE BINOMIAL DISTRIBUTION (PASCAL OR WAITING TIME DISTRIBUTION)

$$X \sim NB(r,p)$$

$$P(X=x)=\binom{r+x-1}{x}p^{r}(1-p)^{x}; \quad x=0,1,\ldots;0\le p\le 1$$

$$E(X)=\frac{r(1-p)}{p} \quad Var(X)=\frac{r(1-p)}{p^{2}}$$

NEGATIVE BINOMIAL DISTRIBUTION

- An alternative form of the pdf:

$$P(Y=y)=\binom{y-1}{r-1}p^{r}(1-p)^{y-r};\ y=r,r+1,\ldots;\quad 0\le p\le 1$$

Note: Y=X+r

$$E(Y)=E(X)+r=\frac{r}{p}\quad Var(Y)=Var(X)=\frac{r(1-p)}{p^{2}}$$

GEOMETRIC DISTRIBUTION

- Distribution of the number of Bernoulli trials required to get the first success.

- Used to model the number of Bernoulli trials needed until the first Success occurs ($P(S)=p$)
  - First Success on Trial 1 $\Rightarrow$ *S*, $y = 1 \Rightarrow p(1)=p$
  - First Success on Trial 2 $\Rightarrow$ *FS*, $y = 2 \Rightarrow p(2)=(1-p)p$
  - First Success on Trial k $\Rightarrow$ *F…FS*, $y = k \Rightarrow p(k)=(1-p)^{k-1}\, p$

- It is the special case of the Negative Binomial Distribution$\rightarrow$ *r=1*.

X~Geometric(p)

$$P(X = x) = p(1-p)^{x-1}, x = 1, 2, \cdots$$

$$E(X) = \frac{1}{p} \quad Var(X) = \frac{(1-p)}{p^2}$$

- Example: If probability is 0.001 that a light bulb will fail on any given day, then what is the probability that it will last at least 30 days?
- Solution:

$$P(X > 30) = \sum_{x=31}^{\infty} 0.001(1-0.001)^{x-1} = (0.999)^{30} = 0.97$$

HYPERGEOMETRIC DISTRIBUTION

- A box contains N marbles. Of these, M are red. Suppose that n marbles are drawn randomly from the box without replacement. The distribution of the number of red marbles, x is

$\boxed{X\text{~Hypergeometric}(\boldsymbol{N},\boldsymbol{M},\boldsymbol{n})}$

$$P(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},x=0,1,\ldots,n$$

It is dealing with finite population.

PRACTICE PROBLEMS

Example 1 As voters exit the polls, you ask a representative random sample of 6 voters if they voted for proposition 100. If the true percentage of voters who vote for the proposition is 55.1%, what is the probability that, *in your sample,* exactly 2 voted for the proposition and 4 did not?

P(2 yes votes exactly) = $\binom{6}{2}(.551)^2\ (.449)^4 = 18.5\%$

Example 2 You are performing a cohort study. If the probability of developing disease in the exposed group is .05 for the study duration, then if you sample (randomly) 500 exposed people, how many do you expect to develop the disease? Give a margin of error (+/- 1 standard deviation) for your estimate.

X ~ binomial (500, .05)
E(X) = 500 (.05) = 25
Var(X) = 500 (.05) (.95) = 23.75
StdDev(X) = square root (23.75) = 4.87
25 ± 4.87

Example 3 Patients arrive at the emergency room of Hospital A at the average rate of 6 per hour on weekend evenings. What is the probability of 4 arrivals in 30 minutes on a weekend evening?

λ = 6/hour = 3/half-hour, x = 4

$$f(4) = \frac{3^4 (2.71828)^{-3}}{4!} = .1680$$

Example 4 Ahmet has removed two dead batteries from his camera and put them into his drawer. In the drawer there are also two good batteries. The four batteries look identical. Ahmet need battery and now randomly selects two of the four batteries. What is the probability he selects the two good batteries?

$$f(x) = \frac{\binom{2}{2}\binom{2}{0}}{\binom{4}{2}} = \frac{\left(\frac{2!}{2!0!}\right)\left(\frac{2!}{0!2!}\right)}{\left(\frac{4!}{2!2!}\right)} = \frac{1}{6} = \quad .167$$

Example 5 At "busy time" a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be on interest to know the number of attempts necessary in order to gain a connection. Suppose that we let $p = 0.05$ be the probability of a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call.

The random variable X is the number of attempts for a successful call. Then X~geometric(0.05),

- So that for with x = 5 and p = 0.05 yields:

- P(X=x) = geometric(5;0.05)
- = $(0.05)(0.95)^4$
- = 0.041
- And the expected number of attempts is $\frac{1}{0.05} = 20$

Example 6 Suppose that a company wishes to hire three new workers and each applicant interviewed has a probability of 0.6 of being found acceptable. What is the probability that exactly six applicants need to be interviewed?

The distribution of the total number of applicants that the company needs to interview Negative Binomial distribution with parameter p = 0.6 and r = 3.

$$P\ X=6\ =\binom{5}{2}\times 0.4^3\times 0.6^3=0.138$$

SOME CONTINUOUS PROBABILITY DISTRIBUTIONS

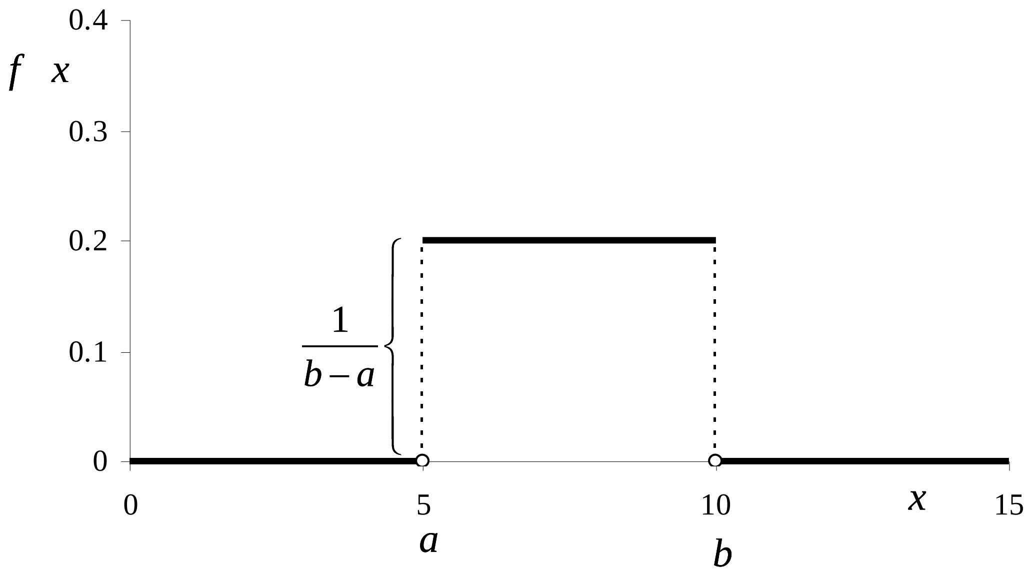
Uniform, Normal, Exponential, Gamma, Chi-Square, Beta Distributions

UNIFORM DISTRIBUTION

A random variable X is said to be uniformly distributed if its density function is

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b.$$

$$\mathrm{E(X)} = \frac{a+b}{2} \qquad V(\mathrm{X}) = \frac{(b-a)^2}{12}$$



Example

- The daily sale of gasoline is uniformly distributed between 2,000 and 5,000 gallons. Find the probability that sales are:
  - Between 2,500 and 3,000 gallons
  - More than 4,000 gallons
  - Exactly 2,500 gallons

$f(x) = 1/(5000-2000) = 1/3000$ for $x: [2000,5000]$

$P(2500 \leq X \leq 3000) = (3000-2500)(1/3000) = .1667$

$P(X \geq 4000) = (5000-4000)(1/3000) = .333$

$P(X=2500) = (2500-2500)(1/3000) = 0$

NORMAL DISTRIBUTION

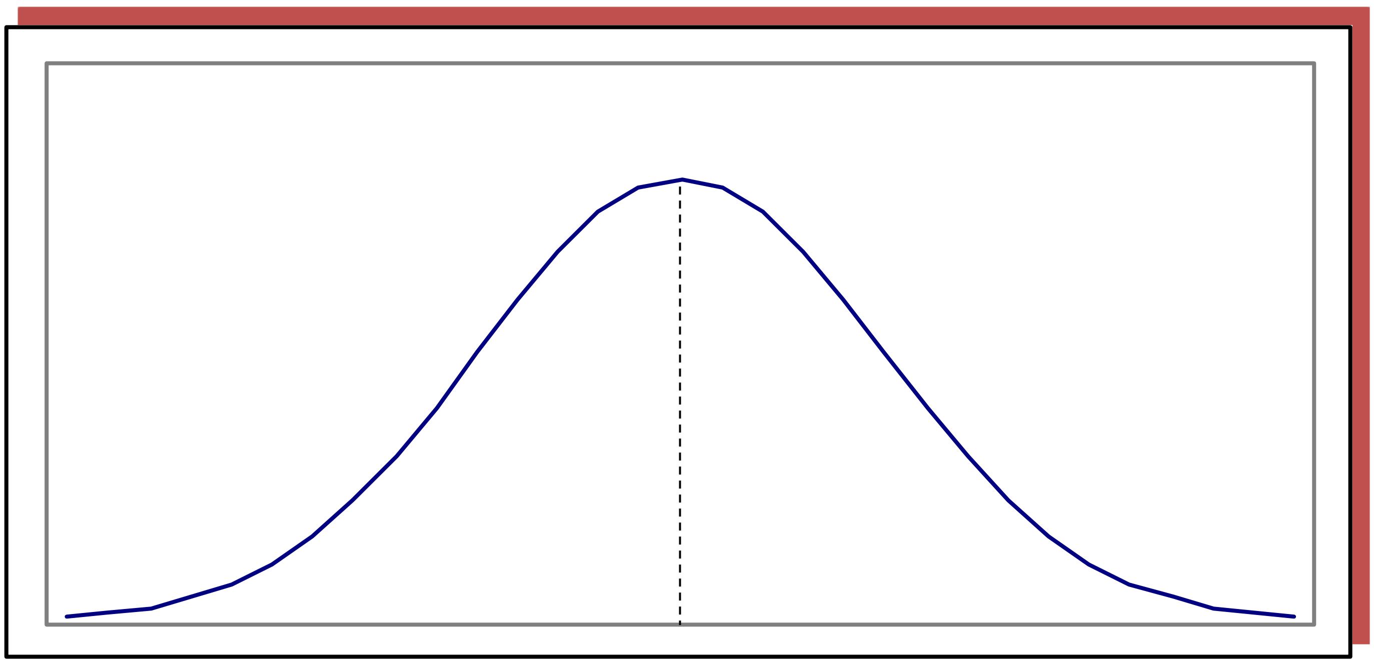
- This is the most popular continuous distribution.
  - Many distributions can be approximated by a normal distribution.
  - The normal distribution is the cornerstone distribution of statistical inference.

- A random variable X with mean μ and variance σ^2 is normally distributed if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \le x \le \infty; \sigma > 0$$

$$where\ \pi = 3.14159\ldots\ \ and\ \ e = 2.71828\ldots$$

THE SHAPE OF THE NORMAL DISTRIBUTION



μ

The normal distribution is bell shaped, and symmetrical around μ.

FINDING NORMAL PROBABILITIES

- Two facts help calculate normal probabilities:
  - The normal distribution is symmetrical.
  - Any normal distribution can be transformed into a specific normal distribution called…

“STANDARD NORMAL DISTRIBUTION”

STANDARD NORMAL DISTRIBUTION

- NORMAL DISTRIBUTION WITH MEAN 0 AND VARIANCE 1.

- IF $X \sim N(\mu, \sigma^2)$, THEN

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

- Example

  The amount of time it takes to assemble a computer is normally distributed, with a mean of 50 minutes and a standard deviation of 10 minutes. What is the probability that a computer is assembled in a time between 45 and 60 minutes?

- Solution
  - If X denotes the assembly time of a computer, we seek the probability P(45<X<60).
  - This probability can be calculated by creating a new normal variable the **standard normal variable.**

By using following transformation.

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

$$P(45<X<60) = P\left(\frac{45 - 50}{10} < \frac{X - \mu}{\sigma} < \frac{60 - 50}{10}\right)$$

$$= P(-0.5 < Z < 1)$$

To complete the calculation we need to compute the probability under the standard normal distribution

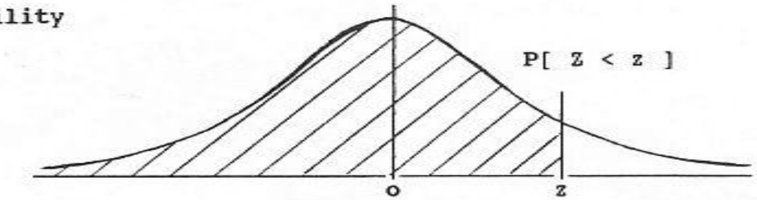
STANDARD NORMAL TABLE 1

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

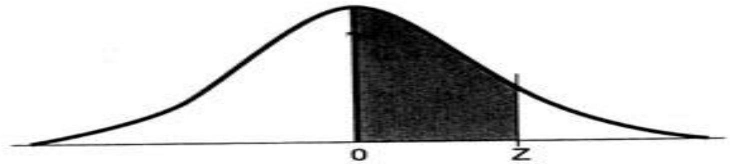
$$P[\, Z < z \,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

STANDARD NORMAL TABLE 2



This table presents the area between the mean and the Z score . When Z=1.96, the shaded area is 0.4750.

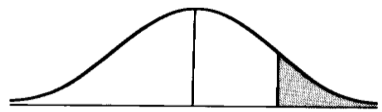
Areas Under the Standard Normal Curve

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000									

Source: Adapted by permission from *Statistical Methods* by George W. Snedecor and William G. Cochran, sixth edition © 1967 by The Iowa State University Press, Ames, Iowa, p. 548.

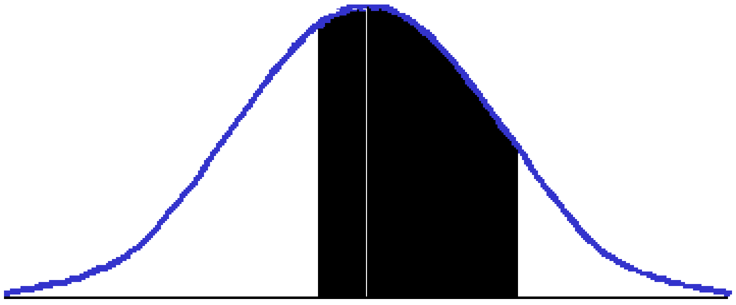
STANDARD NORMAL TABLE 3

TABLE 1 Normal distribution, right-hand tail probabilities



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002

For $P(-.5 < Z < 1)$ We need to find the shaded area



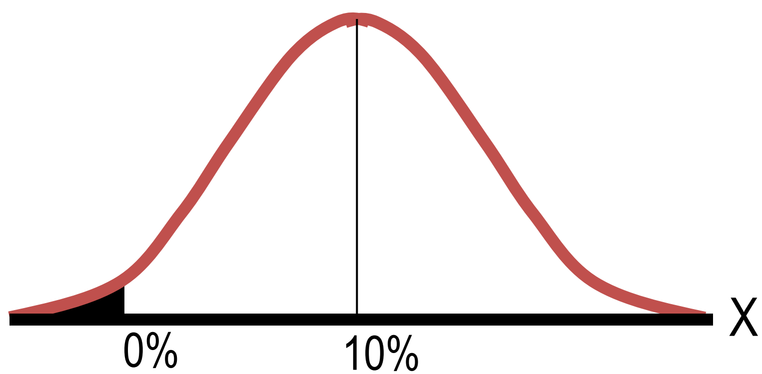
$z_0 = -.5 \quad z_0 = 1$

$= P(-.5<Z<1) = P(-.5<Z<0)+ P(0<Z<1)$

$P(-.5<Z<1) = P(-.5<Z<0)+ P(0<Z<1) = .1915 + .3413 = .5328$

Example

– The rate of return (X) on an investment is normally distributed with a mean of 10% and standard deviation of 5%
– What is the probability of losing money?



$$P(X < 0) = P\left(Z < \frac{0 - 10}{5}\right) = P(Z < -2)$$

$$= P(Z > 2) = 0.5 - P(0 < Z < 2) = 0.5 - .4772 = .0228$$

STANDARDIZATION FORMULA

- If X~N(μ , σ^2), then the standardized value Z of any 'X-score' associated with calculating probabilities for the X distribution is:

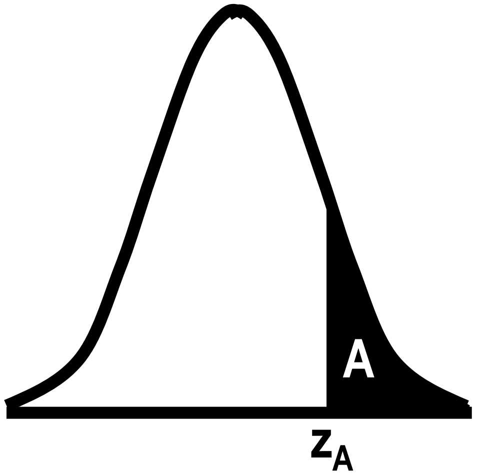
$$Z = \frac{X - \mu}{\sigma}$$

- The standardized value Z of any 'X-score' associated with calculating probabilities for the X distribution is:
- (Converse Formula)

$$x = \mu + z.\sigma$$

FINDING VALUES OF Z

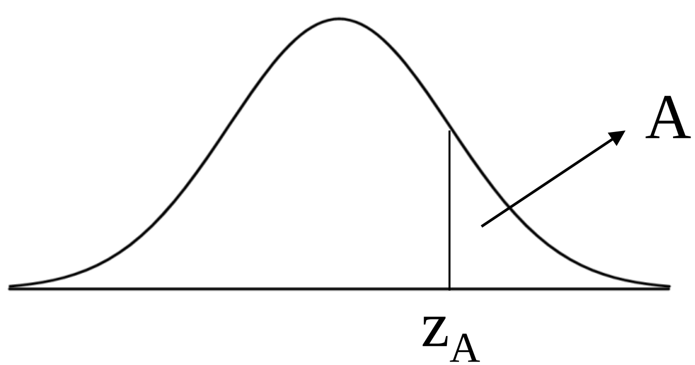
- Sometimes we need to find the value of Z for a given probability
- We use the notation z_A to express a Z value for which $P(Z > z_A) = A$



PERCENTILE

- The p^{th} percentile of a set of measurements is the value for which at most p% of the measurements are less than that value.
- 80^{th} percentile means $P(Z < a) = 0.80$
- If Z ~ N(0,1) and A is any probability, then

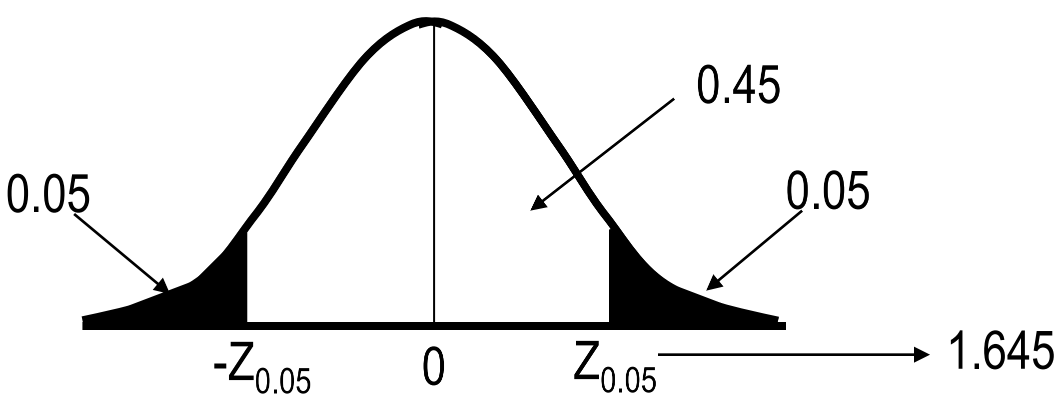
$$P(Z > z_A) = A$$



- Example
  - Determine z exceeded by 5% of the population
  - Determine z such that 5% of the population is below

- Solution

  $z_{.05}$ is defined as the z value for which the area on its right under the standard normal curve is .05.



EXAMPLES

- Let X be rate of return on a proposed investment. Mean is 0.30 and standard deviation is 0.1.

a) $P(X>.55)=?$

b) $P(X<.22)=?$

c) $P(.25<X<.35)=?$

Standardization formula

d) 80^{th} Percentile of X is?

e) 30^{th} Percentile of X is?

Converse Formula

ANSWERS

a) $$P(X > 0.55) = P\left\{\frac{X - 0.3}{0.1} = Z > \frac{0.55 - 0.3}{0.1} = 2.5\right\} = 0.5 - 0.4938 = 0.0062$$

b) $$P(X < 0.22) = P\left\{\frac{X - 0.3}{0.1} = Z < \frac{0.22 - 0.3}{0.1} = -0.8\right\} = 0.5 - 0.2881 = 0.2119$$

c)
$$P(0.25 < X < 0.35) = P\left\{\frac{0.25 - 0.3}{0.1} = -0.5 < \frac{X - 0.3}{0.1} = Z < \frac{0.35 - 0.3}{0.1} = 0.5\right\}$$
$$= 2.*(0.1915) = 0.3830$$

d)
80th Percentile of *X* is $x = \mu + \sigma . z_{0.20} = .3 + (.85)*(.1) = .385$

e)
30th Percentile of *X* is $x = \mu + \sigma . z_{0.70} = .3 + (-.53)*(.1) = .247$

THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

- The normal distribution provides a close approximation to the Binomial distribution when n (number of trials) is large and p (success probability) is close to 0.5.
- The approximation is used **only** when

  $np \geq 5$ and

  $n(1\text{-}p) \geq 5$

- If the assumptions are satisfied, the Binomial random variable X can be approximated by normal distribution with mean μ = np and σ^2 = np(1-p).

- In probability calculations, the continuity correction improves the results. For example, if X is Binomial random variable, then

$$P(X \leq a) \approx P(X<a+0.5)$$

$$P(X \geq a) \approx P(X>a-0.5)$$

EXAMPLE

- Let X ~ Binomial(25,0.6), and want to find P(X ≤ 13).
- Exact binomial calculation:

$$P(X \le 13) = \sum_{x=0}^{13} \binom{25}{x} (0.6)^{x} (0.4)^{25-x} = 0.267$$

- Normal approximation (w/o correction): Y~N(15,2.45²)

$$P(X \le 13) \approx P(Y \le 13) = P(Z \le \frac{13-15}{2.45}) = P(Z \le -0.82) = 0.206$$

Normal approximation is good, but not great!

EXPONENTIAL DISTRIBUTION

- The exponential distribution can be used to model
  - the length of time between telephone calls
  - the length of time between arrivals at a service station
  - the lifetime of electronic components.
- When the number of occurrences of an event follows the Poisson distribution, the time between occurrences follows the exponential distribution.

A random variable is exponentially distributed if its probability density function is given by

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0, \lambda > 0$$

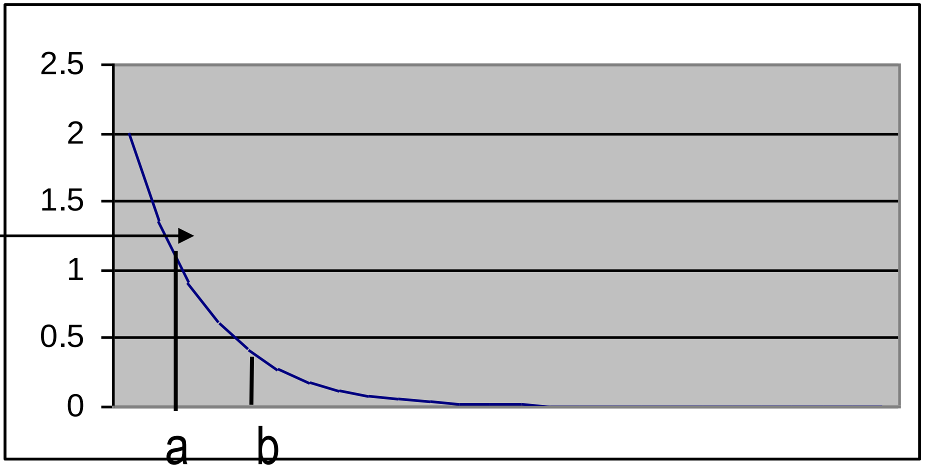
λ is a distribution parameter.

$$E(X) = \lambda \qquad V(X) = \lambda^2$$

The cumulative distribution function is

$$F(x) = 1 - e^{-x/\lambda}, x \geq 0$$

$$\boxed{P(a<X<b) = e^{-a/\lambda} - e^{-b/\lambda}}$$



- Finding exponential probabilities is relatively easy:
  - $P(X < a) = P(X \leq a) = F(a) = 1 - e^{-a/\lambda}$
  - $P(X > a) = e^{-a/\lambda}$
  - $P(a < X < b) = e^{-a/\lambda} - e^{-b/\lambda}$

Example

The service rate at a supermarket checkout is 6 customers per hour.

- If the service time is exponential, find the following probabilities:
  - A service is completed in 5 minutes,
  - A customer leaves the counter more than 10 minutes after arriving
  - A service is completed between 5 and 8 minutes.

- Solution
  - A service rate of 6 per hour
    A service rate of .1 per minute (λ^{-1} = .1/minute).
  - $P(X < 5) = 1\text{-}e^{-.lx} = 1 - e^{-.1(5)} = .3935$
  - $P(X > 10) = e^{-.lx} = e^{-.1(10)} = .3679$
  - $P(5 < X < 8) = e^{-.1(5)} - e^{-.1(8)} = .1572$

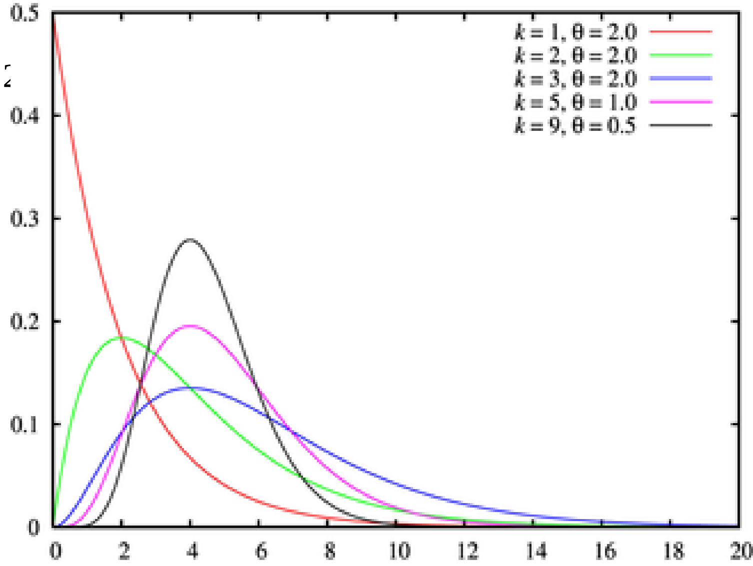
GAMMA DISTRIBUTION

- $X \sim Gamma(\alpha,\beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta},\ x > 0,\ \alpha > 0,\ \beta > 0$$

$$E(X) = \alpha\beta \text{ and } Var(X) = \alpha\beta^2$$

$$M(t) = (1-\beta t)^{-\alpha},\ t < \frac{1}{\beta}$$



- Gamma Function:

$$\Gamma \quad \alpha \quad = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

where α is a positive integer.

Properties:

$$\Gamma \quad \alpha + 1 \quad = \alpha \Gamma \quad \alpha \quad , \alpha > 0$$

$$\Gamma \quad n \quad = \quad n - 1 \ ! \text{ for any integer } n > 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- Let $X_1, X_2, \ldots, X_n$ be independent rvs with $X_i \sim Gamma(\alpha_i, \beta)$. Then,

$$\sum_{i=1}^{n} X_i \sim Gamma \;\; \sum_{i=1}^{n} \alpha_i, \beta$$

- Let X be an rv with $X \sim Gamma(\alpha, \beta)$. Then,

$cX \sim Gamma \;\; \alpha, c\beta$ where c is positive constant.

- Let $X_1, X_2, \ldots, X_n$ be a random sample with $X_i \sim Gamma(\alpha, \beta)$. Then,

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \sim Gamma\left(n\alpha, \frac{\beta}{n}\right)$$

- Special cases: Suppose X~Gamma(α,β)
  - If α=1, then X~ Exponential(β)
  - If α=p/2, β=2, then X~ χ^2 (p) (will come back in a min.)
  - If Y=1/X, then Y ~ inverted gamma.

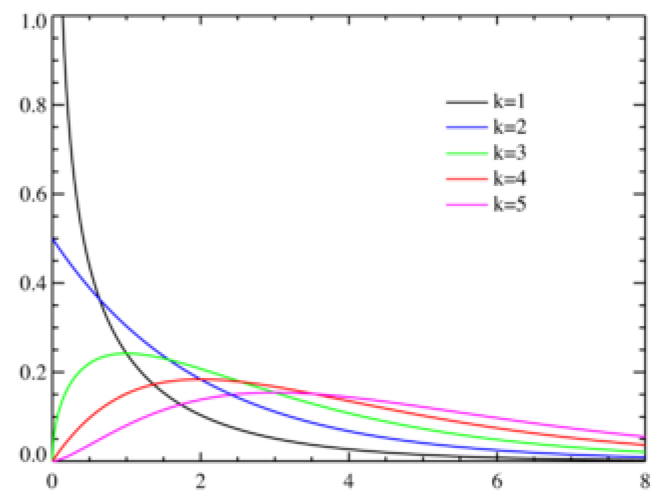
CHI-SQUARE DISTRIBUTION

Chi-square with α degrees of freedom

- $X \sim \chi^2(\alpha) = Gamma(\alpha/2, 2)$

$$f(x) = \frac{1}{2^{\alpha/2}\Gamma(\alpha/2)} x^{\alpha/2-1} e^{-x/2}, x > 0, \alpha = 1,2,\ldots$$

$$E\left(X\right) = \alpha \text{ and } Var\left(X\right) = 2\alpha$$



$$\mathrm{M(t)} = (1 - 2\mathrm{t})^{-\mathrm{p}/2} \quad \mathrm{t} < 1/2$$

DEGREES OF FREEDOM

- In statistics, the phrase ***degrees of freedom*** is used to describe the number of values in the final calculation of a statistic that are free to vary.

- The number of independent pieces of information that go into the estimate of a parameter is called the degrees of freedom (df) .

- How many components need to be known before the vector is fully determined?

- If rv X has $Gamma(\alpha,\beta)$ *distribution, then* $Y=2X/\beta$ *has* $Gamma(\alpha,2)$ *distribution. If* 2α *is positive integer, then* Y *has* $\chi^2_{2\alpha}$ *distribution.*

- Let X be an rv with $X \sim N(0, 1)$. Then,

$$X^2 \sim \chi^2_1$$

- Let $X_1, X_2, \ldots, X_n$ be a r.s. with $X_i \sim N(0,1)$. Then,

$$\sum_{i=1}^{n} X_i^2 \sim \chi^2_n$$

BETA DISTRIBUTION

- The Beta family of distributions is a continuous family on (0,1) and often used to model proportions.

$$f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},\ 0<x<1,\ \alpha>0,\ \beta>0.$$

where

$$B(\alpha,\beta)=\int_0^1 x^{\alpha}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$E(X)=\frac{\alpha}{\alpha+\beta}\ \text{and}\ Var(X)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

CAUCHY DISTRIBUTION

- It is a symmetric and bell-shaped distribution on $(-\infty,\infty)$ with pdf

$$f(x)=\frac{1}{\pi\sigma}\frac{1}{1+(\frac{x-\theta}{\sigma})^2},\ \sigma>0$$

Since $E|X|=\infty$, the mean does not exist.

- The mgf does not exist.
- θ measures the center of the distribution and it is the median.
- If X and Y have N(0,1) distribution, then Z=X/Y has a Cauchy distribution with $\theta=0$ *and* $\sigma=1$.

LOG-NORMAL DISTRIBUTION

- An rv X is said to have the ***lognormal distribution***, with parameters μ and σ^2, if $Y=ln(X)$ has the N(μ, σ^2) distribution.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} x^{-1} e^{-(\ln x-\mu)^2/(2\sigma^2)}, \quad 0 < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0$$

- The lognormal distribution is used to model continuous random quantities when the distribution is believed to be skewed, such as certain income and lifetime variables.

STUDENT'S T DISTRIBUTION

- This distribution will arise in the study of population mean when the underlying distribution is normal.
- Let Z be a standard normal rv and let U be a chi-square distributed rv independent of Z, with ν degrees of freedom. Then,

$$X = \frac{Z}{\sqrt{U/\nu}} \sim t_{\nu}$$

When $n \to \infty$, $X \to N(0,1)$.

F DISTRIBUTION

- Let U and V be independent rvs with chi-square distributions with ν_1 and ν_2 degrees of freedom. Then,

$$X = \frac{U / \nu_1}{V / \nu_2} \sim F_{\nu_1, \nu_2}$$

MOMENT GENERATING FUNCTION

The moment generating function (m.g.f.) of random variable X is defined as

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{\text{all } x} e^{tx} f(x)dx & \text{if } X \text{ is cont.} \\ \sum_{\text{all } x} e^{tx} f(x) & \text{if } X \text{ is discrete} \end{cases}$$

for t Є (-h,h) for some h>0.

PROPERTIES OF M.G.F.

- M(0)=E[1]=1
- If a r.v. X has m.g.f. $M_X(t)$, then Y=aX+b has a m.g.f.

$$e^{bt}M_X(at)$$

- $E(X^k) = M_X{}^{(k)}(0) \quad \textit{where } M_X{}^{(k)} \textit{ is the } k^{th} \textit{ derivative.}$
- M.g.f does not always exists (e.g. Cauchy distribution)

Consider the series expansion of e^x :

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

Note that by taking derivatives with respect to x, we get:

$$\frac{de^x}{dx} = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots = 1 + x + \frac{x^2}{2!} + \ldots = e^x$$

$$\frac{d^2e^x}{dx^2} = 0 + 1 + \frac{2x}{2!} + \ldots$$

Now, Replacing x with tX, we get:

$$e^{tX} = \sum_{i=0}^{\infty} \frac{(tX)^i}{i!} = 1 + tX + \frac{(tX)^2}{2} + \frac{(tX)^3}{6} + \ldots$$

Taking derivatives with respect to t and evaluating at $t = 0$:

$$\left.\frac{de^{tX}}{dt}\right|_{t=0} = 0 + X + \left.\frac{2tX^2}{2!} + \frac{3t^2X^3}{3!} + \ldots\right|_{t=0} = X + tX^2 + \left.\frac{t^2X^3}{2!} + \ldots\right|_{t=0} = X + 0 + 0 + \ldots = X$$

$$\left.\frac{d^2e^{tX}}{dt^2}\right|_{t=0} = 0 + X^2 + tX^3 + \ldots\Big|_{t=0} = X^2 + 0 + \ldots = X^2$$

$$\Rightarrow M_X'(t)\Big|_{t=0} = E(X), \quad M_X''(t)\Big|_{t=0} = E\left(X^2\right), \ \ldots \quad M_X^{(k)}(t)\Big|_{t=0} = E\left(X^K\right)$$

BINOMIAL DISTRIBUTION

$$M(t)=E\quad e^{tX}\quad=\sum_{y=0}^{n}e^{tx}\left[\binom{n}{x}p^{x}(1-p)^{n-x}\right]=$$

$$=\sum_{y=0}^{n}\binom{n}{x}\quad pe^{t}\quad^{x}(1-p)^{n-x}=\quad pe^{t}+(1-p)\quad^{n}$$

$$M'(t)=n\quad pe^{t}+(1-p)\quad^{n-1}pe^{t}=np\quad pe^{t}+(1-p)\quad^{n-1}e^{t}$$

$$M''(t)=np\left[(n-1)\quad pe^{t}+(1-p)\quad^{n-2}pe^{t}\right]e^{t}+\quad pe^{t}+(1-p)\quad^{n-1}\left[e^{t}\right]$$

$$E(\mathrm{X})=M'(0)=np\quad p(1)+(1-p)\quad^{n-1}(1)=np$$

$$E\left(X^{2}\right)=M''(0)=np\left\{\left[(n-1)\left(p(1)+(1-p)\right)^{n-2}p(1)\right](1)+\left(p(1)+(1-p)\right)^{n-1}[1]\right\}=$$

$$=np\left((n-1)p+1\right)=n^{2}p^{2}-np^{2}+np=n^{2}p^{2}+np(1-p)$$

$$V(\mathrm{X})=E\left(X^{2}\right)-\left[E(\mathrm{X})\right]^{2}=n^{2}p^{2}+np(1-p)-(np)^{2}=np(1-p)$$

GEOMETRIC DISTRIBUTION

$$M(t) = E\quad e^{tX} \quad = \sum_{y=1}^{\infty} e^{tx} q^{x-1} p = \frac{p}{q} \sum_{y=1}^{\infty} e^{tx} q^{y} = \frac{p}{q} \sum_{y=1}^{\infty} \quad qe^{t} \quad {}^{x} =$$

$$= \frac{pqe^{t}}{q} \sum_{y=1}^{\infty} \quad qe^{t} \quad {}^{x-1} = \frac{pe^{t}}{1-qe^{t}} = \frac{pe^{t}}{1-(1-p)e^{t}}$$

$$M\,'(t) = \frac{pe^{t}}{(1-(1-p)e^{t})^{2}} \qquad M\,''(t) = \frac{pe^{t}(1+(1-p)e^{t})}{(1-(1-p)e^{t})^{3}}$$

$$E(\mathrm{X}) = M\,'(0) = \frac{1}{p} \qquad E(\mathrm{X}^{2}) = M\,''(0) = \frac{p(1+(1-p))}{(1-(1-p))^{3}} = \frac{(1+(1-p))}{p^{2}}$$

$$Var(X) = E(X^{2}) - \mathrm{E}(\mathrm{X})^{2} = \frac{(1+(1-p))}{p^{2}} - \frac{1}{p^{2}} = \frac{(1-p)}{p^{2}}$$

POISSON DISTRIBUTION

$$M(t) = E \quad e^{tX} \quad = \sum_{y=0}^{\infty} e^{tx} \left[\frac{e^{-\lambda}\lambda^{y}}{\mathrm{x}!} \right] = \sum_{y=0}^{\infty} \frac{e^{-\lambda} \quad \lambda e^{t} \quad ^{x}}{\mathrm{x}!} =$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda e^{t} \quad ^{x}}{\mathrm{x}!} = e^{-\lambda} e^{\lambda e^{t}} = e^{\lambda \; e^{t}-1}$$

$$M\,'(t) = e^{\lambda \; e^{t}-1} \; \lambda e^{t} \qquad M\,''(t) = e^{\lambda \; e^{t}-1} \; \lambda e^{t} + e^{\lambda \; e^{t}-1} \; (\lambda e^{t})^{2}$$

$$E(\mathrm{X}) = \lambda \qquad \mathrm{E}(\mathrm{X}^{2}) = \lambda + \lambda^{2}$$

$$\mathrm{var}(X) = \lambda + \lambda^{2} - \lambda^{2} = \lambda$$

EXPONENTIAL DISTRIBUTION

$$M(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx}\left(\frac{1}{\lambda}e^{-x/\lambda}\right)dx = \frac{1}{\lambda}\int_0^\infty e^{-x\left(\frac{1}{\lambda}-t\right)}dx$$

$$= \frac{1}{\lambda}\int_0^\infty e^{-x\left(\frac{1-t\lambda}{\lambda}\right)}dx = \frac{1}{\lambda}\int_0^\infty e^{-x/\lambda^*}dx \qquad \text{where} \quad \lambda^* = \frac{\lambda}{1-t\lambda}$$

$$M(t) = \frac{1}{-1/\lambda^*}\left(\frac{1}{\lambda}\right)e^{-x/\lambda^*}\Big|_0^\infty = -\frac{\lambda^*}{\lambda}(0-1) = \frac{\lambda^*}{\lambda} = \frac{1}{1-\lambda t} = (1-\lambda t)^{-1}$$

$$M'(t) = -1(1-\lambda t)^{-2}(-\lambda) = \lambda(1-\lambda t)^{-2}$$

$$M''(t) = -2\lambda(1-\lambda t)^{-3}(-\lambda) = 2\lambda^2(1-\lambda t)^{-3}$$

$$E(\mathrm{X}) = M'(0) = \lambda$$

$$V(\mathrm{X}) = M''(0) - \left[M'(0)\right]^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

GAMMA DISTRIBUTION

$$M(t) = E\left[e^{tX}\right] = \int_0^{\infty} e^{tx}\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}\right) dx =$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)} dx = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-x/\beta^*} dx \qquad \text{where } \beta^* = \frac{\beta}{1-\beta t}$$

$$M(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \Gamma(\alpha)\left(\beta^*\right)^{\alpha} = (1-\beta t)^{-\alpha}$$

$$M'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M''(t) = (-\alpha-1)\alpha\beta(1-\beta t)^{-\alpha-2}(-\beta) = \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$$

$$E(\mathrm{X}) = M'(0) = \alpha\beta$$

$$V(\mathrm{X}) = M''(0) - \left[M'(0)\right]^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

NORMAL DISTRIBUTION

$$M(t)=E\left[e^{tX}\right]=\int_{-\infty}^{\infty}e^{tx}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}\right)dx=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}+\frac{x\mu}{\sigma^2}-\frac{\mu^2}{2\sigma^2}+tx\right\}dx=$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}+\frac{\mathrm{x}(\mu+t\sigma^2)}{\sigma^2}-\frac{\mu^2}{2\sigma^2}\right\}dx$$

Completing the square: $(\mu+t\sigma^2)^2=\mu^2+2\mu t\sigma^2+(t\sigma^2)^2$

$$\Rightarrow M(t)=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}+\frac{\mathrm{x}(\mu+t\sigma^2)}{\sigma^2}-\frac{\mu^2}{2\sigma^2}-\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}+\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}\right\}dx=$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}+\frac{\mathrm{x}(\mu+t\sigma^2)}{\sigma^2}-\frac{\mu^2+2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}+\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}\right\}dx=$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{\left[x-(\mu+t\sigma^2)\right]^2}{\sigma^2}\right)+\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}\right\}dx=$$

$$=\exp\left\{\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{\left[x-(\mu+t\sigma^2)\right]^2}{\sigma^2}\right)\right\}dx$$

The last integral being 1, since it is integrating over the density of a normal R.V.: $Y\sim N(\mu+t\sigma^2,\sigma^2)$

$$M(t)=\exp\left\{\frac{2\mu t\sigma^2+(t\sigma^2)^2}{2\sigma^2}\right\}=\exp\left\{\mu t+\frac{t^2\sigma^2}{2}\right\}$$

CHARACTERISTIC FUNCTION

The c.h.f. of random variable X is defined as

$$\phi_X(t) = E(e^{itX}) = \begin{cases} \int\limits_{all\ x} e^{itx} f(x)dx & if\ \ X\ is\ cont. \\ \sum\limits_{all\ x} e^{itx} f(x) & if\ \ X\ is\ discrete \end{cases}$$

for all real numbers t. $i^2 = -1, i = \sqrt{-1}$

C.h.f. always exists.

Examples

- Binomial Distribution: $C(t) = (pe(it) + 1 - p)^n$
- Poisson Distribution: $C(t) = e(\lambda(e(it) - 1))$
- Negative Binomial Dist.: $C(t) = p^k (1 - qe(it))^{-k}$
- Exponential Dist.: $\mathrm{C(t)} = (1 - \frac{it}{\lambda})^{-1}$
- Gamma Dist.: $\mathrm{C(t)} = (1 - \frac{it}{\lambda})^{-r}$
- Normal Dist.: $\mathrm{C}(t) = e(it\mu - \frac{t^2\sigma^2}{2})$
- Chi-sqr Dist.: $C(t) = (1 - 2it)^{-\frac{1}{2}n}$